Solutions

EE 552
Homework 1

1 Simple Binary Tests

1.1 Exponential Random Variables in Queuing

\[ p_X(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}, \quad x \geq 0 \]

\[ H_0 : T \sim \exp(\mu_0). \]

\[ H_1 : T \sim \exp(\mu_1), \quad \mu_1 > \mu_0. \]

1.1.1 Prove that the likelihood ratio test is equivalent to comparing \( T \) to a threshold \( \gamma \).

\[
\text{LRT} = \frac{p_T(t \mid H_1)}{p_T(t \mid H_0)} \overset{H_1}{>}_{H_0} \eta
\]

\[
\frac{\mu_0}{\mu_1} \exp \left\{ t \left( \frac{1}{\mu_0} - \frac{1}{\mu_1} \right) \right\} \overset{H_1}{>}_{H_0} \eta
\]

Since equation 1 is monotone and increasing in \( t \), we can rewrite the LRT as

\[
\text{LRT} : t \overset{H_1}{>}_{H_0} \gamma,
\]

where

\[
\gamma = \frac{\mu_0 \mu_1}{\mu_1 - \mu_0} \ln \left( \frac{\mu_1}{\mu_0} \eta \right)
\]

Thus the likelihood ratio test is equivalent to comparing \( T \) to a threshold \( \gamma \).

1.1.2 Bayes test. Find \( \gamma \) as a function of the costs and the a priori probabilities.

\[
\eta = \frac{P_0(C_{10} - C_{00})}{P_1(C_{01} - C_{11})} = \frac{P_0 C_F}{P_1 C_M}
\]

since \( C_{00} = C_{11} = 0 \). This implies that Bayes test becomes

\[
T \overset{H_1}{<}_{H_0} \frac{\mu_0 \mu_1}{\mu_1 - \mu_0} \ln \left( \frac{\mu_1}{\mu_0} \frac{P_0 C_F}{P_1 C_M} \right)
\]

1.1.3 Assume that a Neyman-Pearson test is used. Find \( \gamma \) as a function of the bound on the false alarm probability \( P_F \), where \( P_F = P(\text{say } H_1 \mid H_0 \text{ is true}) \).

\[
P_F = \int_{-\infty}^{\infty} p_T(t \mid H_0) \, dt = \exp \left[ -\frac{\gamma}{\mu_0} \right]
\]

\[
\gamma = -\mu_0 \ln(P_F)
\]
1.1.4 Plot the ROC for this problem for $\mu_0 = 1$ and $\mu_1 = 5$.

$$P_D = \int_{\gamma}^{\infty} Pr (t \mid H_1) \ dt = \exp \left[ -\frac{\gamma}{\mu_1} \right]$$

(6)

1.1.5 Consider $N$ independent and identically distributed measurements of $T$. Find the likelihood ratio test and probability density function for the likelihood ratio test.

$$T_i : \text{i.i.d., } i = 1, 2, \ldots, N$$

$$P_T (t \mid H_l) = \prod_{i=1}^{N} \frac{1}{\mu_i} e^{-\frac{t}{\mu_i}}$$

$$= \frac{1}{\mu_l^N} \exp \left\{ -\frac{1}{\mu_l} \sum_{i=1}^{N} t_i \right\}, \ l = 0, 1$$

$$\text{LRT} = \frac{Pr (t \mid H_1)}{Pr (t \mid H_0)} \sim H_0 \eta$$

$$= \frac{\mu_0^N}{\mu_1^N} \exp \left\{ \sum_{i=1}^{N} t_i \left( \frac{1}{\mu_0} - \frac{1}{\mu_1} > 0 \right) \right\} \sim H_0 \eta$$

Thus any monotone function of $\sum_{i=1}^{N} t_i$ can be used. Thus the likelihood ratio test is equivalent to testing

$$l (T) = \frac{1}{N} \sum_{i=1}^{N} \frac{t_i}{H_0 \gamma}$$

Let $z = \sum_{i=1}^{N} t_i$. This is an $N$-Erlang random variable and if $y = ax$, then

$$P_Y (y) = \frac{1}{|a|} P_X \left( \frac{x}{a} \right),$$

which implies then that

$$P_x (z) = \frac{e^{-\frac{z}{\mu}}}{\mu (N - 1)!} \left( \frac{1N}{\mu} \right)^{N-1} \mu$$

Figure 1: The ROC curve for section 1.1.4.
Thus we get that

\[ H_0: P_{l(T)}(l \mid H_0) = N \frac{e^{-\frac{1l\mu_0}{\mu_0}}}{\mu_0(N-1)!} \left( \frac{lN}{\mu_0} \right)^{N-1}, l \mid 0 \]

\[ H_1: P_{l(T)}(l \mid H_1) = N \frac{e^{-\frac{1l\mu_1}{\mu_1}}}{\mu_1(N-1)!} \left( \frac{lN}{\mu_1} \right)^{N-1}, l \mid 0 \]

1.2 Gaussian Variance

For a single observation of a zero mean, \( \sigma^2 \) variance Gaussian Random Variable, the pdf is given by

\[ P_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \tag{7} \]

For \( N \) observations, we get

\[ P_R(r) = \prod_{i=1}^{N} \left[ \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{r_i^2}{2\sigma^2}} \right] = \left( \frac{1}{2\pi\sigma^2} \right)^{\frac{N}{2}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{N} r_i^2 \right\}, l = 0, 1 \tag{8} \]

1.2.1 Find the likelihood ratio test

From equation 8, we find that the LRT is equivalent to

\[ \text{LRT} = \frac{P_R(r \mid H_1)}{P_R(r \mid H_0)} \overset{H_1}{\gtrless} \overset{H_0}{\lessgtr} \eta \]

\[ \left( \frac{\sigma_0}{\sigma_1} \right)^N \exp \left\{ -\frac{1}{2} \left( \frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} \right) \sum_{i=1}^{N} r_i^2 \right\} \overset{H_1}{\gtrless} \overset{H_0}{\lessgtr} \eta \]

1.2.2 Simplify the LRT to a comparison with a sufficient statistic

It is easy to see that the LRT is monotone in \( \sum_{i=1}^{N} r_i^2 \). Thus the sample variant, \( l(R) = \frac{1}{N} \sum_{i=1}^{N} R_i^2 \) is sufficient to a threshold.

1.2.3 Find an expression for the probability of false alarm, \( P_F \), and the probability of a miss, \( P_M \).

\[ l(R) = \frac{1}{N} \sum_{i=1}^{N} R_i^2 \]

\[ = \frac{\sigma^2}{N} \sum_{i=1}^{N} \left( \frac{R_i}{\sigma} \right)^2 \]

\[ = \frac{N}{\chi^2_{RV}} \]

\[ Z = \sum_{i=1}^{N} \left( \frac{R_i}{\sigma} \right)^2 \]

\[ P_Z(z) = \frac{z^{N/2} e^{-\frac{z}{2}}}{2^{(N/2)} \Gamma(N/2)}, z > 0 \]
Problem 1.2 (d), ROC Curve

Figure 2: The ROC curve for section 1.2.4.

\[ P_{l(R)}(l) = \frac{N}{\sigma^2} \left( \frac{N-2}{2} \right)^{\sigma^2/2} \exp \left\{ -\frac{N}{2\sigma^2} \right\}, \quad l > 0 \]

\[ P_F = \int_{\gamma}^{\infty} P_{l(R)}(l \mid H_0) \, dl \]

\[ P_M = \int_{-\infty}^{\gamma} P_{l(R)}(l \mid H_1) \, dl \]

1.2.4 Plot the ROC for \( \sigma_0^2 = 1 \), \( \sigma_1^2 = 2 \), and \( N = 2 \).

\[ P_F = \int_{\gamma}^{\infty} e^{-l} \, dl = e^{-\gamma} \]

\[ P_D = \int_{\gamma}^{\infty} \frac{1}{2} e^{-\frac{l}{2}} \, dl = e^{-\frac{\gamma}{2}} \]

since \( N = 2 \)

\[ P_{l(R)} = \frac{1}{\sigma^2} e^{-\frac{l}{2\sigma^2}} \]

1.3 Binary Observations

\[ H_0: \quad P(R_i = head) = 0.5, \quad i = 1, 2, \ldots, N \]

\[ H_1: \quad P(R_i = head) = p, \quad i = 1, 2, \ldots, N \]

\[ p(R) = \binom{N}{k} p^k (1-p)^{N-k}, \quad k = \text{Number of Heads} \]

1.3.1 Determine optimal likelihood ratio test. Show that the number of heads is a sufficient statistic.

\[ \Lambda(R) = \frac{p(R \mid H_1)}{p(R \mid H_0)} \]

\[ \begin{align*}
\eta & < \frac{\nu_1}{H_0} \\
p^k (1-p)^{N-k} & < \eta \left( \frac{1}{2} \right)^N
\end{align*} \]
1.3.2 Plot the ROC for $N = 10$, and $p = 0.7$.

Note, the following equations only hold for $P > 0.5$.

$$P_F = \sum_{k=\lceil \gamma + 1 \rceil}^{N} (0.5)^N = 1 - \sum_{k=0}^{\lfloor \gamma \rfloor} (0.5)^N$$

$$P_D = 1 - P_M = 1 - \sum_{k=0}^{\lfloor \gamma \rfloor} p^k (1 - p)^{N-k}$$

1.3.3 Randomized Testing.

Let the randomized test be setup as shown in Figure 1.3.3, where $\delta = \phi (l)$. Let us define the following notation:

$$\Psi (l) = \begin{cases} 1, & \Lambda (l) > \eta \\ B, & \Lambda (l) = \eta \\ 0, & \Lambda (l) < \eta \end{cases}$$

where B is a Bernoulli Random Variable, $(0 - 1)$, with probability $P (B = 1) = \delta = \phi (l)$. $E_i [\Psi (l)]$ is the expectation of $\Psi (l)$ under $H_i$.

• First: Let $\Psi' (l)$ be any function such that $0 \leq \Psi' (l) \leq 1$ and $E_o [\Psi' (l)] = E_o [\Psi (l)]$. We want to show that $E_1 [\Psi (l)] \geq E_1 [\Psi' (l)]$. Note that

$$\sum_{l} \Delta E \beta (p (l | H_1) - \gamma p (l | H_0)) \geq 0 \quad (9)$$
When $\Psi(l) = 1$, $\Delta \geq 0$, $\beta > 0$. When $\Psi(l) = 0$, $\Delta \leq 0$, $\beta < 0$. When $\Psi(l) = B$, we don’t care about the value of $\Delta$ since $\beta = 0$. Thus we can rewrite equation 9 as

$$E_1[\Psi(l)] \geq E_1[\Psi'(l)]$$

- **Deterministic Case**, $\gamma$ is not an integer. If $L > k$, choose $H_1$. Otherwise, choose $H_0$. Call it “test k.” Note that $k$ is an integer. Thus

$$P_{FA_k} = \sum_{l > k} p(l \mid H_0)$$

$$P_{D_k} = \sum_{l > k} p(l \mid H_1)$$

- **Randomized Case**, $\gamma = k = \text{integer}$.

$$P_{FA} = \sum_{l > \gamma} p(l \mid H_0) + \delta p(k \mid H_0)$$

if $l = \gamma = k$, flip the $\delta$-coin

$$= P_{FA_k} + \delta \left[ P_{FA_{k+1}} - P_{FA_k} \right]$$

Thus

$$P_{FA} = (1 - \delta) P_{FA_k} + P_{FA_{k+1}}$$

$$P_{D} = (1 - \delta) P_{D_k} + \delta P_{D_{k+1}}$$

- We now need to find $\phi$ as a function of $\alpha$.

$$\alpha = P_{FA}$$

$$= P_{FA_k} + \delta p(k \mid H_0)$$

which implies then that

$$\delta = \frac{\alpha - P_{FA_k}}{p(k \mid H_0)}$$

where $k$ is such that $P_{FA_k} \leq \alpha < P_{FA_{k+1}}$.

### 2 Likelihood Ratio as a Random Variable

Prove the following properties when the likelihood ratio, $\lambda(R)$, is given as

$$\Lambda(R) = \frac{p(R \mid H_1)}{p(R \mid H_0)}$$

(11)
2.1 $E[\Lambda^n | H_1] = E[\Lambda^{n+1} | H_0]$

$$E[\Lambda^n | H_1] = \int_{-\infty}^{\infty} \lambda^n P_\Lambda(\lambda | H_1) \, d\lambda$$
$$= \int_{-\infty}^{\infty} \lambda^{n+1} P_\Lambda(\lambda | H_0) \, d\lambda$$
$$= E[\Lambda^{n+1} | H_0]$$

2.2 $E[\Lambda | H_0] = 1$

$$E[\Lambda | H_0] = \int_{-\infty}^{\infty} \lambda P_\Lambda(\lambda | H_0) \, d\lambda$$
$$= \int_{-\infty}^{\infty} \frac{1}{\lambda} P_\Lambda(\lambda | H_1) \, d\lambda$$
$$= 1$$

2.3 $E[\Lambda | H_1] - E[\Lambda | H_0] = \text{var}(\Lambda | H_0)$

$$E[\Lambda | H_1] = \int_{-\infty}^{\infty} \lambda P_\Lambda(\lambda | H_1) \, d\lambda$$
$$= \int_{-\infty}^{\infty} \lambda^2 P_\Lambda(\lambda | H_0) \, d\lambda$$
$$= E[\Lambda^2 | H_1]$$

$$E[\Lambda | H_0] = 1$$

$$E[\Lambda | H_1] - E[\Lambda | H_0] = E[\Lambda^2 | H_1] - [E[\Lambda | H_0]]^2$$
$$= \text{var}(\Lambda | H_0)$$

3 Matlab Problems

Let

$$R = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_k \end{bmatrix}$$

$$\Theta_k = \exp(-k/4)$$

$$S_k = \exp(-\frac{k}{4T})$$

$$R = aS + \Theta + W$$

$H_0$: $a = 0 \rightarrow$ No Signal Sent

$H_1$: $a = l \rightarrow$ Signal Sent

$H_0$: $R = N(\Theta, V)$

$H_1$: $R = N(S + \Theta, V)$

where $V$ is the covariance matrix, $\sigma^2 I_k$. Thus the the likelihood ratio is given by

$$\Lambda(R) = \frac{P_{R|H_1}(R)}{P_{R|H_0}(R)}$$
\[
\frac{(2\pi)^{-\frac{1}{2}}(\det \mathbf{V})^{-1/2} \exp \left\{ -\frac{1}{2\sigma^2} \mathbf{(R - \Theta - aS)}^T \mathbf{(R - \Theta - aS)} \right\}}{(2\pi)^{-\frac{1}{2}}(\det \mathbf{V})^{-1/2} \exp \left\{ -\frac{1}{2\sigma^2} \mathbf{(R - \Theta)}^T \mathbf{(R - \Theta)} \right\}} \sim_{H_0} \begin{cases} \eta \geq \frac{n_2}{n_1} \\
_1 \geq \frac{n_2}{n_1} \end{cases} \]

\[
\exp \left\{ \frac{1}{2\sigma^2} a^T (R - \Theta) - \frac{1}{2\sigma^2} a^2 S^T S \right\} \sim_{H_0} \begin{cases} \eta \geq \frac{n_2}{n_1} \\
_1 \geq \frac{n_2}{n_1} \end{cases}
\]

This function is monotonic in \( \frac{1}{2\sigma^2} a^T (R - \Theta) \). Thus our test statistic becomes

\[
T = \frac{1}{\sigma^2} S^T \left( R - \Theta \right) \sim_{H_0} \begin{cases} \eta \geq \frac{n_2}{n_1} \\
_1 \geq \frac{n_2}{n_1} \end{cases}
\]

This problem is equivalent to

\[
\begin{align*}
H_0: \quad & X_k = W_k \quad k = 1, 2, \ldots, K; W_k \sim \mathcal{N}(0, \sigma^2) \\
H_1: \quad & X_k = \exp \left\{ -\frac{k}{2T} \right\} + W_k
\end{align*}
\]

Thus

\[
\begin{align*}
E[T] &= \frac{1}{\sigma^2} S^T E[R - \Theta] \\
&= \frac{1}{\sigma^2} a S^T S \\
\text{cov}[T] &= E[(T - \mu_T)(T - \mu_T)] \\
&= E \left[ \frac{1}{\sigma^2} S^T \left( R - \Theta - aS \right) \left( R - \Theta - aS \right)^T S \frac{1}{\sigma^2} \right] \\
&= \frac{1}{\sigma^2} S^T S
\end{align*}
\]

Thus the SNR is given as

\[
\frac{E^2[T]}{\text{var}[T]} = \frac{a^2}{\sigma^2} S^T S
\]

\[
= \frac{1}{\sigma^2} \sum_{k=1}^{k} \exp \left\{ -\frac{k}{2T} \right\} \quad (12)
\]

Normalizing equation 12 gives us

\[
T_1 = \frac{T}{\left( \frac{1}{\sigma} \sqrt{S^T S} \right)^{1/2}} \sim \mathcal{N} \left( \frac{a}{\sigma} \sqrt{S^T S}, 1 \right)
\]

Thus we will use \( T_1 \) as a test statistic. Note, we could also use \( S^T R \) as a test statistic.
Figure 5: The ROC curve for section 3.

Figure 6: The ROC curve for section 3.

Figure 7: The ROC curve for section 3.
4 Code

function hw1Solutions()

% Problem 1.1 d

LV_MU_0 = 1;
LV_MU_1 = 5;
LV_PF = [eps:0.0001:1];
LV_GAMMA = -LV_MU_0 .* log(LV_PF);
LV_PD = exp(- LV_GAMMA ./ LV_MU_1);
figure;plot(LV_PF, LV_PD);
title('Problem 1.1 (d), ROC Curve');
xlabel('P_F');
ylabel('P_D');
print(gcf, '-deps', './figures/HW1Problem1.1d.eps');

% Problem 1.2 d

LV_GAMMA = [0:0.1:1000];
LV_PD1 = exp(-LV_GAMMA ./ 2);
LV_PF1 = exp(-LV_GAMMA);
figure;plot(LV_PF1, LV_PD1);
title('Problem 1.2 (d), ROC Curve');
xlabel('P_F');
ylabel('P_D');
print(gcf, '-deps', './figures/HW1Problem1.2d.eps');

% Problem 1.3 b

LV_GAMMA = 1:11;
N = 10;
pd = 0.7;
pf = 0.5;

PD = 1 - binocdf(LV_GAMMA - 1, N, pd);
PF = 1 - binocdf(LV_GAMMA - 1, N, pf);
figure;plot(PF, PD, '^', PF, PD);
title('Problem 1.3 (b), ROC Curve');
xlabel('P_F');
ylabel('P_D');
legend('Deterministic Test', 'Randomized Test', 4);
print(gcf, '-deps', './figures/HW1Problem1.3b.eps');