Solutions
EE552A Spring 2004
Homework 1

1 Simple Binary Tests

1.1 Exponential Random Variables in Queuing

\[ p_X(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}, \ x \geq 0 \]

\[ H_0 : T \sim \text{exp} (\mu_0). \]

\[ H_1 : T \sim \text{exp} (\mu_1), \ \mu_1 > \mu_0. \]

1.1.1 Prove that the likelihood ratio test is equivalent to comparing \( T \) to a threshold \( \gamma \).

\[
\text{LRT} = \frac{p_T(t \mid H_1)}{p_T(t \mid H_0)} \overset{H_1}{\gtrless} \eta \]

\[
= \frac{\mu_0}{\mu_1} \exp \left\{ \frac{1}{\mu_1} - \frac{1}{\mu_0} \right\} \overset{H_0}{\gtrless} \eta \tag{1}
\]

Since equation 1 is monotone (\( \mu_1 > \mu_0 > 0 \)) and increasing in \( t \), we can rewrite the LRT as

\[
\text{LRT} : t \overset{\mu_0 \mu_1}{\gtrless} H_0 \gamma,
\]

where

\[
\gamma = \frac{\mu_0 \mu_1}{\mu_1 - \mu_0} \ln \left( \frac{\mu_1}{\mu_0} \right) \tag{3}
\]

Thus the likelihood ratio test is equivalent to comparing \( T \) to a threshold \( \gamma \).

1.1.2 Bayes test. Find \( \gamma \) as a function of the costs and the a priori probabilities.

\[
\eta = \frac{P_0 (C_{10} - C_{00})}{P_1 (C_{01} - C_{11})} = \frac{P_0 C_F}{P_1 C_M}, \tag{4}
\]

since \( C_{00} = C_{11} = 0 \). This implies that Bayes test becomes

\[
T \overset{\mu_0 \mu_1}{\gtrless} H_0 \frac{\mu_0 \mu_1}{\mu_1 - \mu_0} \ln \left( \frac{\mu_1}{\mu_0} \frac{P_0 C_F}{P_1 C_M} \right) \tag{5}
\]
1.1.3 Assume that a Neyman-Pearson test is used. Find $\gamma$ as a function of the bound on the false alarm probability $P_F$, where $P_F = P(\text{say } H_1 \mid H_0 \text{ is true})$.

$$P_F = \int_{-\infty}^{\gamma} P_F(t \mid H_0) \, dt = \exp \left[ -\frac{\gamma}{\mu_0} \right]$$

$$\gamma = -\mu_0 \ln(P_F)$$

1.1.4 Plot the ROC for this problem for $\mu_0 = 1$ and $\mu_1 = 5$.

$$P_D = \int_{-\infty}^{\gamma} P_F(t \mid H_1) \, dt = \exp \left[ -\frac{\gamma}{\mu_1} \right] \quad (6)$$

1.1.5 Consider $N$ independent and identically distributed measurements of $T$. Find the likelihood ratio test and probability density function for the likelihood ratio test.

$$T_i: \text{i.i.d., } i = 1, 2, \ldots, N$$

$$P_T(t \mid H_l) = \prod_{i=1}^{N} \frac{1}{\mu_l} e^{-\frac{t_i}{\mu_l}}$$

$$= \frac{1}{\mu_l^N} \exp \left\{ -\frac{1}{\mu_l} \sum_{i=1}^{N} t_i \right\}, \quad l = 0, 1$$

$$\text{LRT} = \frac{P_T(t \mid H_1)}{P_T(t \mid H_0)} \overset{H_1}{\underset{H_0}{\sim}} \eta$$

$$= \frac{\mu_0^N}{\mu_1^N} \exp \left\{ \sum_{i=1}^{N} t_i \left( \frac{1}{\mu_0} - \frac{1}{\mu_1} \right) \right\}_{H_1} \overset{H_0}{\underset{H_0}{\sim}} \eta$$

Thus any monotone function of $\sum_{i=1}^{N} t_i$ can be used. Thus the likelihood ratio test is equivalent to testing

$$l(T) = \frac{1}{N} \sum_{i=1}^{N} T_i \overset{H_1}{\underset{H_0}{\sim}} \gamma$$
Let \( z = \sum_{i=1}^{N} t_i \). This is an \( N\text{-Erlang} \) random variable and if \( y = ax \), then
\[
P_Y(y) = \frac{1}{|a|} P_X\left(\frac{x}{a}\right),
\]
which implies then that
\[
P_Z(z) = e^{-\frac{z}{\mu}} \left(\frac{LN}{\mu}\right)^{N-1}
\]
Thus we get that
\[
H_0: P_{l(r)}(l \mid H_0) = N \frac{e^{-\frac{\ln}{\mu_0}}}{\mu_0 (N-1)!} \left(\frac{LN}{\mu_0}\right)^{N-1}, \ l > 0
\]
\[
H_1: P_{l(r)}(l \mid H_1) = N \frac{e^{-\frac{\ln}{\mu_1}}}{\mu_1 (N-1)!} \left(\frac{LN}{\mu_1}\right)^{N-1}, \ l > 0
\]

### 1.2 Gaussian Variance

For a single observation of a zero mean, \( \sigma^2 \) variance Gaussian Random Variable, the pdf is given by
\[
P_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \tag{7}
\]
For \( N \) observations, we get
\[
P_R(r) = \prod_{i=1}^{N} \left[ \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{r^2}{2\sigma_i^2}} \right]
\]
\[
= \left(\frac{1}{2\pi\sigma_i^2}\right)^N \exp \left\{ -\frac{1}{2\sigma_i^2} \sum_{i=1}^{N} r_i^2 \right\}, \ l = 0, 1 \tag{8}
\]

#### 1.2.1 Find the likelihood ratio test

From equation 8, we find that the LRT is equivalent to
\[
\text{LRT} = \frac{P_R(r \mid H_1)}{P_R(r \mid H_0)} \overset{\eta_1}{\overset{\eta_0}{\gtrless}} H_0
\]
\[
\left(\frac{\sigma_0}{\sigma_1}\right)^N \exp \left\{ \frac{1}{2} \left( \frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} \right) \sum_{i=1}^{N} r_i^2 \right\} \overset{\eta_1}{\overset{\eta_0}{\gtrless}} H_0
\]

#### 1.2.2 Simplify the LRT to a comparison with a sufficient statistic

It is easy to see that the LRT is monotone in \( \sum_{i=1}^{N} R_i^2 \). Thus the sample variant, \( l(R) = \frac{1}{N} \sum_{i=1}^{N} R_i^2 \) is sufficient to a threshold.

#### 1.2.3 Find an expression for the probability of false alarm, \( P_F \), and the probability of a miss, \( P_M \).

\[
l(R) = \frac{1}{N} \sum_{i=1}^{N} R_i^2
\]
Figure 2: The ROC curve for section 1.2.4.

\[ P_F = \int_{\gamma}^{\infty} \frac{e^{-l}}{\sigma^2} dl = e^{-\gamma} \]

\[ P_D = \int_{\gamma}^{\infty} \frac{1}{2} e^{-\frac{l}{2\sigma^2}} dl = e^{-\frac{\gamma}{2}} \]

since \( N = 2 \)

\[ P_{l(R)} = \frac{1}{\sigma^2} e^{-\frac{l}{\sigma^2}} \]

1.3 Binary Observations

\( H_0: \ P(R_i = \text{head}) = 0.5, \quad i = 1, 2, \ldots, N \)
\[ H_1: \quad P(R_i = \text{head}) = p, \quad i = 1, 2, \ldots, N \]
\[ p(R) = \binom{N}{k} p^k (1-p)^{N-k}, \quad k = \text{Number of Heads} \]

1.3.1 Determine optimal likelihood ratio test. Show that the number of heads is a sufficient statistic.

\[ \Lambda(R) = \frac{p(R \mid H_1)}{p(R \mid H_0)} \]
\[ p^k (1-p)^{N-k} \]
\[ \begin{array}{c c c}
\frac{\mu_1 >}{H_0} & \eta \\
\frac{\mu_1 <}{H_0} & \eta \left(\frac{1}{2}\right)^N \\
\end{array} \]
\[ k \ln p - k \ln (1-p) \]
\[ \begin{array}{c c c}
\frac{\mu_2 >}{H_0} & \gamma, \quad p > 0.5 \\
\frac{\mu_2 <}{H_0} & \gamma, \quad p < 0.5 \\
\end{array} \]

1.3.2 Plot the ROC for \( N = 10 \), and \( p = 0.7 \).

Note, the following equations only hold for \( P > 0.5 \).

\[ P_F = \sum_{k=\lceil \gamma + 1 \rceil}^{N} \binom{N}{k} (0.5)^N = 1 - \sum_{k=0}^{\lfloor \gamma \rceil} \binom{N}{k} (0.5)^N \]
\[ P_D = 1 - P_M = 1 - \sum_{k=0}^{\lfloor \gamma \rceil} \binom{N}{k} p^k (1-p)^{N-k} \]

1.3.3 Randomized Testing.

Let the randomized test be setup as shown in Figure 1.3.3, where \( \delta = \phi(l) \). Let us define the following notation:

\[ \Psi(l) = \begin{cases} 
1, & \Lambda(l) > \eta \\
B, & \Lambda(l) = \eta \\
0, & \Lambda(l) < \eta 
\end{cases} \]

where \( B \) is a Bernoulli Random Variable, \( (0 - 1) \), with probability \( P(B = 1) = \delta = \phi(l) \). \( E_i [\Psi(l)] \) is the expectation of \( \Psi(l) \) under \( H_i \).

- First: Let \( \Psi'(l) \) be any function such that \( 0 \leq \Psi'(l) \leq 1 \) and \( E_o [\Psi'(l)] = E_o [\Psi(l)] \). We want to show that \( E_1 [\Psi(l)] \geq E_1 [\Psi'(l)] \). Note that

\[ \sum_{l} \Delta \left[ \Psi(l) - \Psi'(l) \right] \left[ p(l \mid H_1) - \gamma p(l \mid H_0) \right] \geq 0 \quad (9) \]

When \( \Psi(l) = 1, \Delta \geq 0, \beta > 0 \). When \( \Psi(l) = 0, \Delta \leq 0, \beta < 0 \). When \( \Psi(l) = B \), we don’t care about the value of \( \Delta \) since \( \beta = 0 \). Thus we can rewrite equation 9 as

\[ E_1 [\Psi(l)] \geq E_1 [\Psi'(l)] \quad (10) \]
• Deterministic Case, $\gamma$ is not an integer. If $L > k$, choose $H_1$. Otherwise, choose $H_0$. Call it “test $k$.” Note that $k$ is an integer. Thus

\[ P_{FA_k} = \sum_{l>k} p(l \mid H_0) \]
\[ P_{D_k} = \sum_{l>k} p(l \mid H_1) \]

• Randomized Case, $\gamma = k = \text{integer}$.

\[ P_{FA} = \sum_{l>\gamma} p(l \mid H_0) + \underbrace{\delta p(k \mid H_0)}_{\text{if } l=\gamma=k, \text{flip the coin}} \]
\[ = P_{FA_k} + \delta [P_{FA_{k+1}} - P_{FA_k}] \]

Thus

\[ P_{FA} = (1-\delta) P_{FA_k} + P_{FA_{k+1}} \]
\[ P_D = (1-\delta) P_{D_k} + \delta P_{D_{k+1}} \]

• We now need to find $\phi$ as a function of $\alpha$.

\[ \alpha = P_{FA} \]
\[ = P_{FA_k} + \delta p(k \mid H_0) \]

which implies then that

\[ \delta = \frac{\alpha - P_{FA_k}}{p(k \mid H_0)} \]

where $k$ is such that $P_{FA_k} \leq \alpha < P_{FA_{k+1}}$.

2 Likelihood Ratio as a Random Variable

Prove the following properties when the likelihood ratio, $\lambda(R)$, is given as

\[ \Lambda(R) = \frac{p(R \mid H_1)}{p(R \mid H_0)} \] (11)
2.1 $E[\Lambda^n | H_1] = E[\Lambda^{n+1} | H_0]$ 

$E[\Lambda^n | H_1] = \int_{-\infty}^{\infty} \lambda^n P_{\Lambda}(\lambda | H_1) \, d\lambda$

$= \int_{-\infty}^{\infty} \lambda^{n+1} P_{\Lambda}(\lambda | H_0) \, d\lambda$

$= E[\lambda^{n+1} | H_0]$

2.2 $E[\Lambda | H_0] = 1$

$E[\Lambda | H_0] = \int_{-\infty}^{\infty} \lambda P_{\Lambda}(\lambda | H_0) \, d\lambda$

$= \int_{-\infty}^{\infty} \frac{1}{\lambda} P_{\Lambda}(\lambda | H_1) \, d\lambda$

$= 1$

2.3 $E[\Lambda | H_1] - E[\Lambda | H_0] = \text{var}(\Lambda | H_0)$

$E[\Lambda | H_1] = \int_{-\infty}^{\infty} \lambda P_{\Lambda}(\lambda | H_1) \, d\lambda$

$= \int_{-\infty}^{\infty} \lambda^2 P_{\Lambda}(\lambda | H_0) \, d\lambda$

$= E[\lambda^2 | H_1]$

$E[\Lambda | H_0] = 1$

$= [E[\Lambda | H_0]]^2$

$E[\Lambda | H_1] - E[\Lambda | H_0] = E[\lambda^2 | H_1] - [E[\Lambda | H_0]]^2$

$= \text{var}(\Lambda | H_0)$

3 Matlab Problems

Let

$$R = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_k \end{bmatrix}$$
\[ \Theta_k = \exp(-k/4) \]
\[ S_k = \exp\left(\frac{-k}{4T}\right) \]
\[ R = aS + \Theta + W \]

\[ H_0: \ a = 0 \rightarrow \text{No Signal Sent} \]
\[ H_1: \ a = l \rightarrow \text{Signal Sent} \]
\[ H_0: \ R = \mathcal{N}(\Theta, V) \]
\[ H_1: \ R = \mathcal{N}(S + \Theta, V), \]

where \( V \) is the covariance matrix, \( \sigma^2 I_k \). Thus the the likelihood ratio is given by

\[ \Lambda(R) = \frac{P_{R|H_1}(R)}{P_{R|H_0}(R)} \]

\[ \eta \supset H_0 \eta \]

\[ \frac{(2\pi)^{-\frac{k}{2}}(\det V)^{-1/2}\exp\left\{-\frac{1}{2\sigma^2} (R-\Theta-aS)^T(R-\Theta-aS) \right\}}{(2\pi)^{-\frac{k}{2}}(\det V)^{-1/2}\exp\left\{-\frac{1}{2\sigma^2} (R-\Theta)^T(R-\Theta) \right\}} \]

\[ \exp\left\{\frac{1}{2\sigma^2} aS^T(R-\Theta) - \frac{1}{2\sigma^2} a^2 S^T S \right\} \]

\[ \eta \supset H_0 \eta \]

This function is monotonic in \( \frac{1}{2\sigma^2} aS^T(R-\Theta) \). Thus our test statistic becomes

\[ T = \frac{1}{\sigma^2} S^T (R-\Theta) \]

\[ \supset H_0 \gamma \]

This problem is equivalent to

\[ H_0: \ X_k = W_k \quad k = 1, 2, \ldots, K; W_k \sim \mathcal{N}(0, \sigma^2) \]
\[ H_1: \ X_k = \exp\left\{-\frac{k}{2T}\right\} + W_k \]

Thus

\[ E[T] = \frac{1}{\sigma^2} S^T E[R-\Theta] \]

\[ = \frac{1}{\sigma^2} aS^T S \]

\[ \text{var}[T] = E\left[(T - \mu_T)(T - \mu_T)^T\right] \]

\[ = E\left[\frac{1}{\sigma^2} S^T (R - \Theta - aS)(R - \Theta - aS)^T S \frac{1}{\sigma^2}\right] \]

\[ = \frac{1}{\sigma^2} S^T S \]

Thus the SNR is given as

\[ \frac{E^2[T]}{\text{var}[T]} = \frac{a^2}{\sigma^2} S^T S \]

\[ = \frac{1}{\sigma^2} \sum_{k=1}^{k} \exp\left\{-\frac{k}{2T}\right\} \quad (12) \]
Normalizing gives us

\[ T_1 = \frac{T}{(\frac{1}{\sigma} S^T S)^{1/2}} \sim \mathcal{N}\left(\frac{a}{\sigma} \sqrt{S^T S}, 1\right) \]

Thus we will use \( T_1 \) as a test statistic. Note, we could also use \( S^T R \) as a test statistic.
4 Code

function hw1Solutions()
% Preconditions: None
% Postconditions: None
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Problem 1.1 d
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
LV_MU_0 = 1;
LV_MU_1 = 5;
LV_PF = [eps:0.0001:1];
LV_GAMMA = -LV_MU_0 .* log(LV_PF);
LV_PD = exp(- LV_GAMMA ./ LV_MU_1);
figure;plot(LV_PF, LV_PD);
title('Problem 1.1 (d), ROC Curve');
xlabel('P_F');
ylabel('P_D');
print(gcf, '-deps', './figures/HW1Problem1.1d.eps');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Problem 1.2 d
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
LV_GAMMA = [0:0.1:1000];
LV_PD1 = exp(-LV_GAMMA ./ 2);
LV_PF1 = exp(-LV_GAMMA);
figure;plot(LV_PF1, LV_PD1);
title('Problem 1.2 (d), ROC Curve');
xlabel('P_F');
ylabel('P_D');
print(gcf, '-deps', './figures/HW1Problem1.2d.eps');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Problem 1.3 b
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

Figure 7: The ROC curve for section 3.
LV_GAMMA = 1:11;
N = 10;
pd = 0.7;
pf = 0.5;

PD = 1 - binocdf(LV_GAMMA - 1, N, pd);
PF = 1 - binocdf(LV_GAMMA - 1, N, pf);

figure;plot(PF, PD, '^', PF, PD);
title('Problem 1.3 (b), ROC Curve');
xlabel('P_F');
ylabel('P_D');
legend('Deterministic Test', 'Randomized Test', 4);
print(gcf, '-deps', './figures/HW1Problem1.3b.eps');