Alternating Minimization Problems for Transmission Tomography Using Energy Detectors

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Outline

- Transmission Tomography
- ML Version of Transmission Tomography
- Alternating Minimization Algorithm
- Energy Detectors
- Alternating Minimization Algorithm
- Conclusions
Transmission Tomography Mean Data Model

• Line integrals through object
• Survival probabilities: mean, exponential attenuation
• Mean number of counts (integral and discrete versions) (integration time, current…)
  – Source-detector pair $y$ in $Y$
  – Energy indexed by $E$

\[
q(y, E) = I_0(y, E) \exp \left( - \int h(y \mid x) \mu(x, E) dx \right)
\]

\[
q(y, E) = I_0(y, E) \exp \left( - \sum_{x \in X} h(y \mid x) \mu(x, E) \right)
\]
Mean Data Model

- X-ray signal detector energy spectrum
- Computational
- 120 kVp
- 30 cm PMMA (plexiglass)
- \( I_0(y,E) \) proportional to current, time
Transmission Tomography Detectors

- Poisson model, mean $q_m$
- Energy detector model

$$d_m = \int_{y \in Y_m} \int_{E \in E_m} N(dy, dE), \quad P(d_m = n) = \frac{q_m^n}{n!} e^{-q_m}$$

$$d_m = \int_{y \in Y_m} \int_{E \in E_m} EN(dy, dE), \quad E\{\exp(j \nu d_m)\} = ...$$
Poisson Data Loglikelihood

• ML Problem: maximize $l_{cd}$ over admissible $q(y,E) \leftarrow$ exponential family

$$l_{cd}(\mu) = \sum_{m} \int_{y \in Y_m} \int_{E \in E_m} \ln q(y, E) N(dy, dE)$$

$$- \sum_{m} \int_{y \in Y_m} \int_{E \in E_m} q(y, E) dy dE$$

$$q(y, E) = I_0(y, E) \exp \left( - \sum_{x \in X} h(y \mid x) \mu(x, E) \right)$$
Constituent Model

- Regularization part 1: sum of known constituents
  - Polystyrene plus CaCl solution
- Practice: space-varying by tissue
- Complexity: $I$ forward/backward projections

\[ \mu(x, E) = \sum_{i=1}^{I} c_i(x) \mu_i(E) \]
Poisson Data Loglikelihood

- ML Problem: maximize $l_{cd}$ over admissible $q(y,E) \leftrightarrow$ exponential family

$$q(y, E) = I_0(y, E) \exp \left( - \sum_{x \in X} h(y \mid x) \sum_{i=1}^{I} c_i(x) \mu_i(E) \right)$$

- Variational representation of $l_{cd}$
- Convex decomposition lemma:

$$\ln \left( \sum_{i} a_i \right) = - \min_{\pi \in \mathcal{P}} \left( \sum_{i} \pi_i \ln \frac{\pi_i}{a_i} \right)$$
Alternating Minimization Algorithm

\[ l_{cd}(\mu) = -\min_{q \in \mathbb{E}} \min_{p \in \mathcal{L}} I(p\|q) \]

\[ I(p\|q) = \sum_m \int_{y \in Y_m} \int_{E \in \mathbb{E}_m} p(y, E) \ln \frac{p(y, E)}{q(y, E)} dy dE \]

\[ -\sum_m \int_{y \in Y_m} \int_{E \in \mathbb{E}_m} [q(y, E) - p(y, E)] dy dE \]
X-ray Signals

X-ray flux \( \propto I \cdot t \);

Detector converts flux to signal:

\[
Signal = It \int_{0}^{kVp} g(E)Q(E)dE
\]

For counter (conventional),

\( g(E) = 1 \), \( S = Q_{tot} \), pdf is Poisson

120kVp, 30cm PMMA

Detector energy spectrum \( Q(E) \)
Mini CT, AM Iteration 0000001

No Ordered Subsets

22 Ordered Subsets

132 Ordered Subsets

David G. Politte
October 31, 2002
Mini CT, AM Iteration 0000002

No Ordered Subsets

22 Ordered Subsets

132 Ordered Subsets

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Mini CT, AM Iteration 0000100

No Ordered Subsets  

22 Ordered Subsets  

132 Ordered Subsets  

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Mini CT, AM Iteration 0010000

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Mini CT, AM Iteration 0100000

No Ordered Subsets

22 Ordered Subsets

132 Ordered Subsets

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Mini CT, AM Iteration 1000000

No Ordered Subsets

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132 Ordered Subsets

\[
\begin{align*}
\text{Log Error} &= C_1 \cdot \log(\text{Iteration Number}) + C_2
\end{align*}
\]
Energy Integrator

For energy integrator, \( g(E) = E \),

\[
Signal = \int_{0}^{kVp} EQ(E) dE
\]

The pdf is a compound Poisson distribution given by
(Whiting SPIE Medical Imaging 2002 vol 4682 pp 53-60):

\[
pdf(S) = e^{-\lambda \tau} \sum_{k=0}^{\infty} \frac{\left(\frac{\lambda S}{c}\right)^k}{k!}
\]
Experimental Measurements

Measured and Modeled NEQ=40
Experiments

- Plexiglass Cylinder in air PMMA=32cm
- Fit mean baseline to sinogram
- Collect histograms around nominal values
- Fit histogram using a one parameter family
Conclusions

• Algorithms for Transmission Tomography
  – Difficulty of maximization
  – Implementations

• Energy detectors
  – Increased problem complexity
  – Implementation issues

• Future work: carefully address
  – Need for enhanced model
  – Feasibility of implementations
\[ d_m = \int_{y \in \mathcal{Y}_m} \int_{E \in \mathcal{E}_m} N(dy, dE) \]

\[ d_m = \int_{y \in \mathcal{Y}_m} \int_{E \in \mathcal{E}_m} EN(dy, dE) \]

\[ d_m = \int_{y \in \mathcal{Y}_m} \int_{E \in \mathcal{E}_m} N(dy, dE) \quad P(d_m = n) = \frac{q^m_n}{n!} e^{-q_m} \]

\[ d_m = \int_{y \in \mathcal{Y}_m} \int_{E \in \mathcal{E}_m} EN(dy, dE) \quad \mathbb{E}\{\exp(\imath \nu d_m)\} = \]
Model for Transmission CT

- $E$: Energies ranging from 19-120 keV
- $I_0(y, E)$: Mean number of source photons
- $\mu(x, E)$: attenuation function; the image we are trying to estimate

$$g(y) = \sum_E I_0(y, E) \exp \left\{ - \sum_x h(y | x) \mu(x, E) \right\} + \sigma(y)$$
Forward Projection

\[ m(y,E) = \sum_x h(y \mid x) \mu(x,E) \]

- \( x \) indexes the image pixel
- \( y \) indexes the source-detector pair, \((\beta, \gamma)\)
- \( \mu(x,E) \) = attenuation value
- \( h(y \mid x) \) = point spread function, average path length of X-ray beam
- \( m(y,E) \) = linear projection of attenuation
Filtered Back Projection

- Linear model assumes scanner measures the attenuation sum of beam.
- Good approximation for objects of slowly varying densities.
- When high-density objects are introduced, nonlinearities (e.g. beam hardening, scatter, noise) are more prominent.
Image Formation

- Optimization Problem: \( \min_g I(d \mid\mid g) \)
  - \( d(y) \) is the measured data
  - \( g(y) \) is our estimate of the data (using our model)
  - \( I(d \mid\mid g) \) is the \( I \)-divergence, an information-theoretic measure to quantify the discrepancy between \( d(y) \) and \( g(y) \).
- Equivalent to maximizing the likelihood
- Equivalent to \( \min_q \min_p I(p\mid\mid q) \)
Alternating Minimization Algorithm

\[ q^{(k)}(y, E) = I_0(y, E) \exp(-\sum h(y|x)m^{(k)}(x, E)) \]

\[ p^{(k)}(y, E) \]

\[ \frac{1}{Z(x)} \ln \frac{\hat{\theta}(x)}{\tilde{\theta}(x)} \]

\[ S \]

\[ m^{(k+1)}(x, E) \]

\[ Z(x) \]

Results of Algorithm

FBP

AM
Incorporating Prior Knowledge

We can include in the image reconstruction prior knowledge of (for example):

- Attenuation coefficients, geometry of metal objects (hip prostheses, brachytherapy applicators, dental fillings, prostate seeds, etc.)

Need to adapt AM algorithm to:

- Determine the location & orientation (pose) of metal object
- Reconstruct only the unknown pixels
Creating synthetic data

$$\mu_t(x,E) = \mu(E)c(x)$$

$$a(y) = \sum_{E} I_0(y,E) \exp \left\{ - \sum_{x} h(y \mid x) \mu_t(x,E) \right\}$$

Forw Proj., etc.

I_0(y,E)

Poisson noise generator

$$d(y)$$

$$\sigma(y)$$
Objects at known pose

- Metal object at pose \( \theta = (x_1, x_2, \varphi) \)
- Partition pixel space \( X \) into “metal pixels” \( X_m(\theta) \) and surrounding pixels \( X_s(\theta) = X - X_m(\theta) \).
- Image reconstruction:
  \[
  \mu(x, E) = \mu_m, x \square X_m(\theta) \\
  = \mu^{(k+1)}(x, E), x \square X_s(\theta)
  \]
Partial volumes

- Metal object at pose $\theta=(x_1, x_2, \phi)$
- Partition pixel space $X$ into:
  - metal pixels $X_m(\theta)$
  - surrounding pixels $X_s(\theta)$
  - edge pixels $X_e(\theta)$
- Image reconstruction:
  \[ \mu(x, E) = \mu_m, x \square X_m(\theta) \]
  \[ = \mu^{(k+1)}(x, E), x \square X_s(\theta) \]
  \[ \geq \alpha \mu_m, x \square X_e(\theta) \]
Objects at unknown pose

- **Goal:** Pose Search—concurrently update image estimate & pose $\theta$

- **Method:**
  - Form $\mu(x,E;\theta)$, by putting rods into $\mu^{(k+1)}(x,E)$ at a candidate pose $\theta$
  - Select the $\mu(x,E;\theta)$ and $\theta$ that minimizes $I(p||q)$ as the next image and pose

Error = (0.003 mm, 0.005 mm, 0.015°)
Next Steps

- Incorporate regularization into model
- Perform experiments on real data that investigate pose search techniques
- Further investigate partial volume effects
- Explore other methods of performing the pose search
Conclusions

• Developed a model for transmission tomography incorporating physical effects
• Demonstrated an AM algorithm derived from this model
• Improved upon the model when prior information is included
• Algorithm can simultaneously update the image while performing a parametric search